AD 730123

Nonlinear Evolution and Saturation of an Unstable Electrostatic Wave

B. D. Fried, C. S. Liu
R. W. Means, and R. Z. Sagdeev

August, 1971 PPG-93

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

Plasma Physics Group
Department of Physics
University of California
Los Angeles, California 90024

D D C
RECEIVED
SEP 23 1971
C

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
Springfield, Va. 22151

This work was partially supported by the Office of Naval Research, Grant #N00014-69-A-0200-4023; The National Science Foundation, Grant #GP-22817; the Atomic Energy Commission, Contract AT(04-3)-34, Project #157; and the National Aeronautics and Space Administration, Contract #NGL-05-007-190.

15

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (*Corporate author*)	2a. REPORT SECURITY CLASSIFICATION
	2b. GROUP

3. REPORT TITLE

Nonlinear Evolution and Saturation of an Unstable Electrostatic Wave

4. DESCRIPTIVE NOTES (*Type of report and inclusive dates*) Technical Report

5. AUTHOR(S) (*First name, middle initial, last name*)

B.D. Fried, C.S. Liu, R.W. Means, and R.Z. Sagdeev

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August, 1971	14	5

8a. CONTRACT OR GRANT NO. N00014-69-A-0200-4023	9a. ORIGINATOR'S REPORT NUMBER(S) PPG-93
b. PROJECT NO. NR012-306	
c.	9b. OTHER REPORT NO(S) (*Any other numbers that may be assigned this report*)
d.	

10. DISTRIBUTION STATEMENT

Unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	ONR Arlington, Virginia 22217

13. ABSTRACT

The nonlinear development and saturation of a single Langmuir wave driven unstable by a gentle bump in the tail of the distribution function in a collisionless plasma is studied by treating the resonant particles numerically. Over a wide range of parameter values, the amplitude of the potential ϕ is found to saturate at such a level that the ratio $g \equiv \omega_b/\gamma_o \approx 3.2$, where $\omega_b = (ek^2\phi/m)^{12}$ is the bounce frequency of the trapped particles in the wave trough and γ_o is the linear growth rate, approximately given by the classical Landau value. In view of the importance of inverse Landau damping for many instabilities, this work should have wide applicability and the results should be suitable for direct experimental tests.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

Nonlinear Evolution and Saturation of an Unstable Electrostatic Wave§

B. D. Fried,* C. S. Liu,† R. W. Means, and R. Z. Sagdeev††

University of California, Los Angeles, California 90024

Abstract

The nonlinear development and saturation of a single Langmuir wave driven unstable by a gentle bump in the tail of the distribution function in a collisionless plasma is studied by treating the resonant particles numerically. Over a wide range of parameter values, the amplitude of the potential ϕ is found to saturate at such a level that the ratio $g \equiv \omega_b/\gamma_o \simeq 3.2$, where $\omega_b = (ek^2\phi/m)^{1/2}$ is the bounce frequency of the trapped particles in the wave trough and γ_o is the linear growth rate, approximately given by the classical Landau value. In view of the importance of inverse Landau damping for many instabilities, this work should have wide applicability and the results should be suitable for direct experimental tests.

---

* Also at TRW Systems.

† Present address: Institute for Advanced Studies, Princeton, New Jersey.

†† Permanent address: Institute of High Temperature, USSR Academy of Science, Moscow, USSR.

§ This work was partially supported by the Office of Naval Research, Grant #N00014-69-A-0200-4023; the National Science Foundation, Grant #GP-22817; the Atomic Energy Commission, Contract AT(04-3)-34, Project #157; and the National Aeronautics and Space Administration, Contracts NGR-05-007-190 and NGR-05-007-116.

We report here the result of a study of the evolution and saturation of a Langmuir wave driven unstable by the resonant particles whose velocities are near the wave phase velocity with a "gentle bump-in-tail" distribution. The method used represents a new approach to computer stimulation of a plasma in that only a small fraction of the particles, the resonant particles (i.e., trapped and nearly trapped particles in the nonlinear stage), are followed numerically, the remainder being treated analytically.(1) In the linear theory, the interaction of the wave with the resonant particles in the bump leads to the well-known Landau growth of the wave.(2)

As the unstable wave grows, the nonlinear effects become important. If the spectrum of unstable waves is sufficiently wide, then the nonlinear development can be adequately described by the quasi-linear theory,(3) in which the trapping of the particles by the wave field is neglected. If, on the other hand, the spectrum of the unstable modes is narrow, either because the linear growth rate is a peaked function of wave number or due to initial conditions, then a single mode dominates and the trapping of the resonant particles is frequently the most important nonlinear effect.

The effects of the trapped particles on the damping of a single large amplitude Langmuir wave have been previously studied with analytic methods.(4) In these analytic treatments, however, it is assumed that both the amplitude and the phase of the wave are constant in time ($\gamma_o/\omega_b \ll 1$). With these approximations, many aspects of the nonlinear wave-particle interaction which are important in the case of unstable waves, such as the effect of transition between the trapped states and untrapped states (trapping and detrapping) and the adiabatic heating and cooling of the trapped particles due to the amplitude variation, are neglected. In the present work, we follow the evolution of an unstable wave by treating the dynamics of the resonant particles numerically; it is thus possible to allow both the amplitude and the phase of the wave to vary in time, as dictated by Poisson's equation, i.e., to study the complementary case $\gamma_o/\omega_{bo} \gtrsim 1$, where ω_{bo} is ω_b at $t = 0$. The nonlinear phase shift, varying on the same time scale as the amplitude, is particularly significant in affecting the nonlinear evolution and saturation as well as in conserving the energy in the wave frame. Neglecting the phase shift in the nonlinear stage, as assumed in a recent work,[5] leads to an erroneous conclusion on saturation.

The bulk of the electrons (nonresonant particles) are adequately described by the usual linear approximation, even at the time of saturation, so they appear simply via a dielectric constant which modifies the field produced by the resonant particles. Their thermal velocity is assumed to be so small compared with the phase velocity of the wave, $U = \omega_p/k$, that they can be approximated by a cold fluid with dielectric function

$$\varepsilon(\omega,k) = 1 - (\omega_p/\omega)^2 \quad . \tag{1}$$

For the resonant particles, we assume a distribution uniform in space and linear in velocity over a range $-\delta v \leq v-U \equiv w \leq \delta v$:

$$f_R(v) = \begin{cases} (\alpha/2\delta v)(1+\beta w/\delta v) & \text{for } |w| \leq \delta v \\ 0 & \text{for } |w| > \delta v \end{cases} \tag{2}$$

where w is the velocity in the initial wave frame; α is the fraction of the resonant particles compared to the bulk electrons; β is the fractional change in f from $v = U$ to $v = U \pm \delta v$; and δv is typically chosen to be of order 10 (γ_o/k) so that there remain untrapped resonant particles in the nonlinear stage. This "single-wave approximation" is valid for such a broad gentle bump, where one would expect many modes to grow, if a single wave with amplitude sufficiently above the thermal noise level is launched into the system initially, with electric field $E = E_o \sin kx$ and a corresponding perturbation in the charge density of the nonresonant particles: $\rho_{NR}(x) = \rho_o(0) \cos kx$ where $\rho_o(0) = kE_o/4\pi$. The evolution of the wave in time is described by Poisson's equation

$$\varepsilon(\omega,k)\, E(\omega,k) = 4\pi[\rho_o(\omega,k) + \rho_R(\omega,k)]/ik \tag{3}$$

where ρ_R is the charge density of the resonant particles. Transforming back to the time domain and using Eq. (1), we have

$$E(k,t) = (E_o/2i)\, e^{-i\omega_p t} + (4\pi/ik)[\rho_R(t) + (\omega_p/2i)\, e^{-i\omega_p t} \int_0^t dt\, e^{i\omega_p t'} \rho_R(t')] \tag{4}$$

where the solution with phase velocity $-U$ has been dropped because it cannot have significant interaction with the resonant particles which have $v \simeq U$. The charge density of the resonant particles is given by

$$\rho_R(t) = (\alpha nq/N) \sum_{j=1}^{N} e^{-ikx_j(t)} \tag{5}$$

where n is the total particle density and $x_j(t)$ is the instantaneous position of the j^{th} resonant particle. It is convenient to express $x_j(t)$ in terms of the deviations, z_j, from the unperturbed orbit of the j^{th} particle with initial velocity $U + w_j$: $kx_j = \omega_p t + \psi_j + z_j$ where $\psi_j = k(x_{jo} + w_j t)$ is the unperturbed phase in the initial wave frame. Since the time scale for the energy transfer between the wave and resonant particles, typically of the order of the inverse growth rate or bounce frequency, is much longer than the plasma period ω_p^{-1} , it is convenient to remove the explicit factor $\exp(-i\omega_p t)$ from Eqs. (4), (5), by setting $\rho_R(t) = \exp(-i\omega_p t)\, \rho(t)$ and $E(k,t) = F(E_o/2i) \exp(-i\omega_p t)$. Note that F is in general complex, $F(t) = |F| \exp[i\phi(t)]$ with $F(0) = 1$. Choosing the initial bounce time ω_{bo}^{-1}, as a unit of time, k^{-1} as a unit of distance, we have the following dimensionless Poisson equation:

$$\left(2iE(k,t)/E_{\neg}\right) e^{i\omega_p t} \equiv F(t) = 1 + (2\alpha U^2/N) \left[S(t) + (U/2i) \int_0^t dt' \, S(t')\right] \tag{6}$$

with

$$S(t) = \sum_{j=1}^{N} e^{-i(\psi_j + z_j)} \quad . \tag{7}$$

The equation of motion is

$$\ddot{z}_j = (F/2i) \, e^{i(\psi_j + z_j)} + \text{c.c.} \tag{8}$$

The initial conditions for (8) are $z_j(0) = \dot{z}_j(0) = 0$. To simulate the distribution function of the resonant particles, Eq. (2), we choose the ψ_{jo} to be N_z uniformly spaced points on the interval $(0,2\pi)$: $\psi_{\ell o} = 2\pi\ell/N_z$, $0 \leq \ell \leq (N_z - 1)$. For each $\psi_{\ell o}$, we choose N_v values of w_j on the interval $(-\delta v, \delta v)$, selected to correspond to the resonant particle distribution function, Eq. (2):

$$w_j/\delta v = [(1 - 1/\beta)^2 + 4j/\beta N_v]^{1/2} - 1/\beta \quad . \tag{9}$$

It will be shown in the following that this distribution of $N = N_z N_v$ discrete particles does give approximately the same linear growth rate and frequency shift as the continuum distribution, Eq. (2).

Linear Theory

For sufficiently early times, we can linearize not only the nonresonant particle motion but also that of the resonant particles, whereupon (6) and (8) lead to the usual multi-beam dispersion relation

$$\varepsilon_1(\nu) = 1 - (\alpha U^3/2N_v\nu) \sum_j{}'(\nu - w_j)^{-2} = 0 \tag{10}$$

where the sum is carried out only over the N_v different values of w_j given by Eq. (9).

In the continuum limit ($N_v \to \infty$), corresponding to the choice (2) for f_R, the integration over w in (10) gives

$$\varepsilon_2(\nu) = 1 - \frac{2\alpha U^3}{\nu\delta v} \left\{ \frac{\nu\beta + \delta v}{\nu^2 - (\delta v)^2} + \frac{\beta}{\delta v} \log \frac{\nu - \delta v}{\nu + \delta v} \right\} = 0 \tag{11}$$

If $\delta v \ll \nu$, we expand the terms in brackets in powers of $\delta v/\nu$, obtaining to leading order the usual single beam dispersion relation. This case has recently been studied by O'Neil et al.[1] with an approach similar to ours. In the limit $\delta v \gg \nu$, we can expand in powers of $\nu/\delta v$, obtaining

$$\varepsilon_3(\nu) = 1 - \frac{2\alpha U^3}{\nu(\delta v)^2} \left\{ 1 - i\,\frac{\pi\beta}{2} + \frac{2\nu\beta}{\delta v} + \mathcal{O}[(\nu/\delta v)^3] \right\} \quad . \tag{12}$$

The only root of $\varepsilon_3 = 0$ is

$$\nu = \gamma_L(i - 2/\pi\beta)[1 + 4\gamma_L/\pi\delta v]^{-1} \tag{13}$$

where

$$\gamma_L = \frac{\pi}{2} U^2 f_o'(U)\,\omega_p = \pi\alpha\beta U^3/4(\delta v)^2 \tag{14}$$

is just the Landau value (in our dimensionless units) with $f_o'(U)$ computed from Eq. (2).

We see from Eq. (13) that the growth rate is approximately the Landau value provided $\gamma_L/\delta v \ll 1$. The deviation of γ_o from the Landau value γ_L and the nonvanishing real part of ν are due to the sharp-edged distribution, Eq. (2). This sharp edge can be removed by joining smoothly the distribution of the resonant particles to that of nonresonant particles. This introduces additional "boundary terms" in Eqs. (3) and (6) and, in the linearized case, yields the exact Landau growth rate (14), with vanishing Re ν.

The near vanishing of $\dot{\phi}$ at early times in Fig. 2 is a consequence of the fact that our numerical results are based upon a modified form of (6) which includes these "boundary terms". If they are omitted, then a non-vanishing $\dot{\phi}$, of order γ_L, is obtained in the initial stages, in agreement with (13). Aside from this non-zero $\dot{\phi}$ and the small changes in the linear growth rate, however, these boundary terms do not affect the nonlinear saturation level. It is determined by the actual linear growth rate alone, given by either Eq. (13) or Eq. (14), corresponding to the cases with or without the sharp edge in the distribution.

We return now to the dispersion Eq. (12) for N_v "beams". In general, we expect to find an unstable root, ν_j, with real part between each adjacent pair of w_j, as illustrated in Fig. 1 for a particular choice of parameter values. The associated residue values, $[\epsilon_1'(\nu_\ell)]^{-1}$ are also shown. The continuum value for this case, obtained from Eq. (13), is indicated; it falls very close to the fastest growing root of the N_v-beam modes. The latter also has the largest residue $[\epsilon_1'(\nu_\ell)]^{-1}$, and hence dominates the linear behavior, leading to a good agreement with the continuum results.

Nonlinear Results

We put (6) and (8), including "boundary terms", into standard difference equation form and solve by straightforward step-ahead in time. The wave initially grows at the growth rate γ_o given by Eq. (16) until a time t_{NL} at which the orbit of the particles trapped at the trough of the wave becomes nonlinear. The time for the onset of nonlinearity, t_{NL}, is approximately given by $\int_0^{t_{NL}} dt\, \omega_b(t) \simeq \pi$ or $\omega_b(t_{NL}) \simeq \pi\gamma_o/2 + 1$. After t_{NL} is reached, the growth rate begins to decrease from the linear value, turns negative, and then oscillates about zero with the instantaneous bounce frequency, as shown in Fig. 2. The amplitude of the electric field is initially small, i.e., $g(t=0) = \omega_{bo}/\gamma_o < 1$. It increases exponentially, with growth rate γ_o , until t_{NL}, and then saturates at such a level that the bounce frequency of the trapped particle in the wave trough is approximately equal to 3.2 times the linear growth rate, i.e., $g = \omega_b/\gamma_o = 3.2$. This ratio g at saturation is independent of γ_o and other parametric values so long as $g_o < 1$, as shown in Table I. The nonlinear frequency shift is of the same order of magnitude as the nonlinear growth rate.

In conclusion, we note that

1) Treating numerically only the resonant particles is an efficient simulation technique for any problem where a small fraction of particles exchange energy with waves.

2) The universal character ($g \simeq 3.2$) of the saturation level should be suitable for experimental test.

3) Since many other instabilities involve the same wave-particle resonance mechanism as Landau's problem, similar results are to be expected there as well.

Footnotes and References

1. A similar approach is used by T. M. O'Neil, J. H. Winfrey, and J. H. Malmberg, Phys. Fluids 14, 1204 (1971) in a study of beam-plasma interaction. Of course, instabilities of this character are qualitatively different from the Landau type studied here.

2. L. D. Landau, J. Phys. (USSR) 10, 25 (1946).

3. V. E. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nucl. Fusion 1, 82 (1961); W. E. Drummond and D. Pines, Ann. Phys. (N.Y.) 28, 478 (1964).

4. T. M. O'Neil, Phys. Fluids 8, 2255 (1965); L. M. Al'tshul and V. I. Karpman, Soviet Phys.-JETP 20, 1043 (1965).

5. I. N. Onishchenko, A. R. Linetskii, N. G. Matsiborko, V. D. Shapiro, and V. I. Shevchenko, JETP. Pis. Red. 12, 407 (1970) [Englist transl.: Soviet Phys.-JETP Letters, 281 (1971)]. The height of the first maximum in Fig. 1 of that paper (E vs. t) corresponds to a value of g approximately 80% of that found here, but at later times they show E dropping by a factor of 2. More importantly, if $\dot{\phi}$ is neglected, then saturation is obtained only when the parameters are so chosen that all resonant particles are, in fact, trapped at the time of saturation, a circumstance which invalidates the entire premise of this method (linearization of non-resonant particles). Otherwise, a quite different nonlinear behavior of E(t) is observed; typically, the wave "dies", i.e., $E(t) \to 0$. In all of the cases reported here, the final saturation amplitude is such that an appreciable fraction of the resonant particles remains untrapped. We are indebted to T. M. O'Neil for calling to our attention the importance of including variations in the phase.

Figure Captions

Fig. 1. a) Locus, in the complex ω plane, of roots, ν_ℓ, of the multi-beam dispersion Eq. (10) for $N_v = 30$, with other parameters equal to those in Table I, Case A. The vertical lines show the locations of the 30 beams. The point designated with an arrow is the continuum value of ν given by (13).

b) Relative values of the residues, $[\varepsilon_1'(\nu_\ell)]^{-1}$ for the roots of (1) shown in a). As in a), the second highest point is the residue obtained from the continuum form, (12), for ε.

Fig. 2. Temporal evolution of the instantaneous frequency, $\omega_b(t) = [eE(t)k/m]^{1/2}$; growth rate γ_j, and frequency shift, $\dot{\phi}$, for case A of Table I, with $N_v = 960$ and $N_z = 4$. The vertical scale is in the unit of initial bounce frequency, ω_{bo}; the horizontal scale is in the unit of ω_{bo}^{-1}. The Landau value, γ_L, given by (14), is also shown.

Table I. Summary of parameters and saturation amplitudes for various cases. $\omega_{bo} = (ekE_o/m)^{1/2}$ is the initial bounce frequency, ω_{bf} is the bounce frequency at saturation, γ_L is the Landau growth rate, γ_o is the actual initial growth rate.

Case	A	B	C	D	E	F
α	2.10^{-9}	10^{-7}	4.10^{-9}	10^{-7}	5.10^{-8}	$2.5.10^{-8}$
β	0.5	1.0	0.5	0.1	0.2	0.4
$\Delta=k\delta v/\omega_{bo}$	40	100	40	80	80	80
$U=\omega_p/\omega_{bo}$	2.10^4	10^4	2.10^4	2.10^4	2.10^4	2.10^4
γ_L/ω_{bo}	3.93	7.85	7.85	9.82	9.82	9.82
γ_o/ω_{bo}	3.6	7.4	6.9	8.8	9.0	9.0
ω_{bf}/γ_o	3.23	3.28	3.18	3.18	3.22	3.22

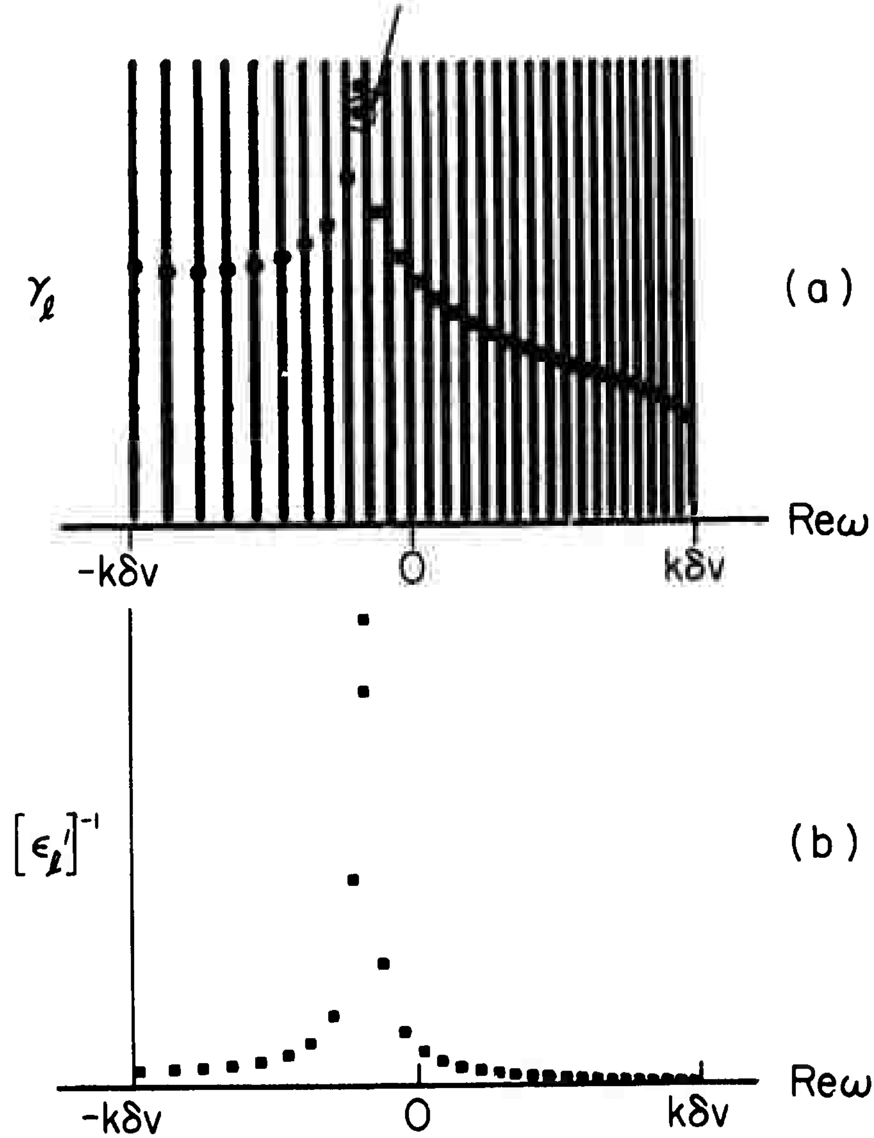

Fig. 1. a) Locus, in the complex ω plane, of roots, ν_ℓ, of the multi-beam dispersion Eq. (10) for $N_v = 30$, with other parameters equal to those in Table I, Case A. The vertical lines show the locations of the 30 beams. The circled point designated with an arrow, is the continuum value of ν given by (13). b) Relative values of the residues, $[\epsilon_1'(\nu_\ell)]^{-1}$ for the roots of (1) shown in a). As in a), the second highest point is the residue obtained from the continuum form, (12), for ϵ.

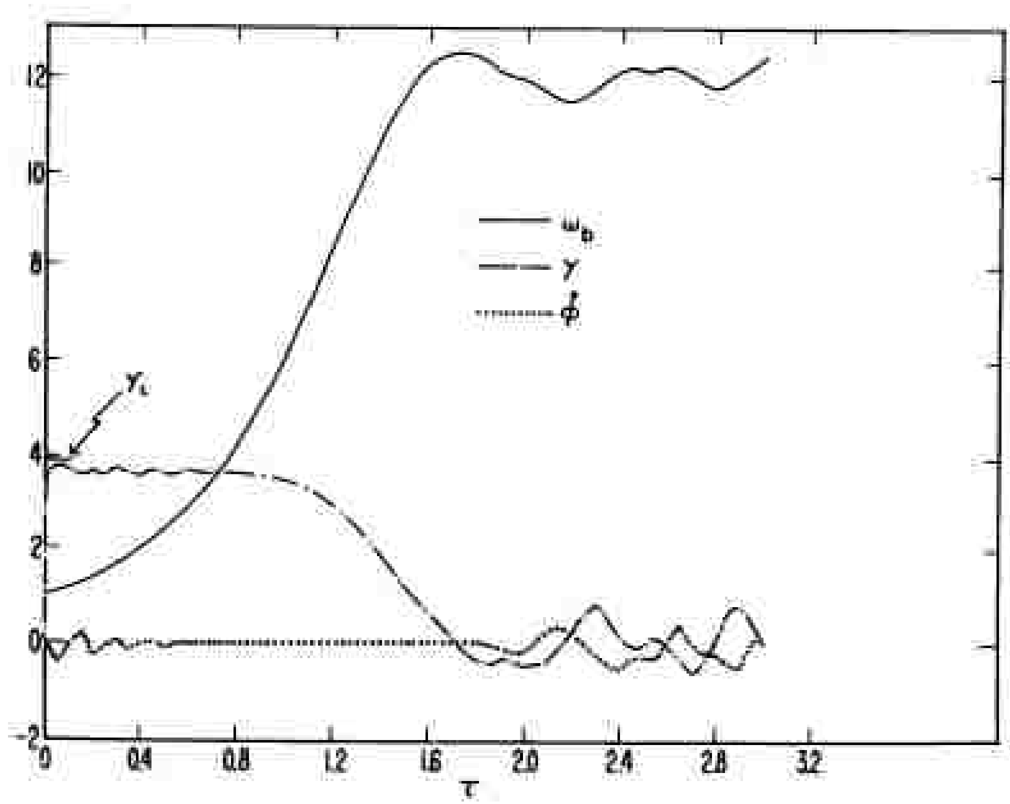

Fig. 2. Temporal evolution of the instantaneous frequency, $\omega_b(t) = [eE(t)k/m]^{1/2}$; growth rate γ_j; and frequency shift, $\dot{\phi}$; for case A of Table I, with $N_v = 960$ and $N_z = 4$. The vertical scale is in the unit of initial bounce frequency, ω_{bo}; the horizontal scale is in the unit of ω_{bo}^{-1}. The Landau value, γ_L, given by (14), is also shown.